1 True or False (10 points)

1. The stability property of a constant equilibrium will not be altered by shifting the equilibrium point to the origin.

2. $\lim_{t\to\infty} \dot{f}(t) = 0$ implies that $f(t)$ has a limit as $t \to \infty$, and vice versa.

3. If $G(s)$ is a transfer function of a Hurwitz, minimum phase LTI SISO system with relative degree equals to one, then $G(s)$ must be SPR.

4. The function $f(x) = x^2$ is not global Lipschitz since its derivative is unbounded.

5. For a linear time invariant system, the dimension of its state space representation always equals to the order of its transfer function.

6. For indirect MRAC method with normalized adaptive law, since the control and adaptive laws are designed independently, the stability analysis can also be done separately.

7. Consider a gradient-based adaptive law

$$\dot{\theta} = \gamma\epsilon\phi, \quad \theta \in \mathbb{R}^n$$

   One necessary condition for θ converging to the true value θ^* is that ϕ must to be sufficiently rich of order n.

8. The equilibrium of a nonlinear system might be unstable, even if its linearized counterpart is stable or uniformly stable.

9. For a linear time invariant system, if the equilibrium is globally asymptotically stable, then it also be globally exponentially stable, furthermore, the dynamic matrix A must be Hurwitz.

10. For the control problem that cannot be solved by pole-placement control method when the plant model is known, one might consider to solve it via **adaptive** pole-placement control method if the strongly coprime assumption is satisfied.

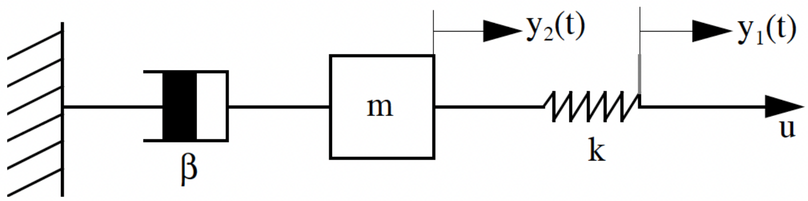
2 Stability Theorem (20 points)

Consider a system described by

$$\begin{aligned}\dot{x}_1 &= -ax_1 + x_1x_2 \\ \dot{x}_2 &= bx_2 - x_1x_2\end{aligned} \tag{1}$$

with $x_1, x_2 \in \mathbb{R}$, a, b are non-zero constants. Define $x = [x_1\, x_2]^\top$.

1. Calculate the equilibria of System (1).
2. Assume $a = b = 1$. Transform System (1) by shifting its nonzero equilibrium to the origin, that is, write down the dynamic equation of $y := x - x^* \in \mathbb{R}^2$ with x^* representing the nonzero equilibrium.
3. Study the stability (in the sense of Lyapunov) of the equilibrium $y^* = [0\,,\,0]^\top$ of the y-system (the system you just developed in question 2) via Lyapunov indirect (first) method and its phase portrait.
4. Prove the following proposition via Lyapunov direct(second) theorem:
   "The equilibrium $x^* = [0\,,\,0]^\top$ of System (1) is unstable if $a < 0, b > 0$"



3 Parameter Estimation and Feedback Controller Design (30 points)

Consider a mass-spring-damper system shown in Fig above whose dynamic can be described by

$$
\begin{aligned}
k(y_1 - y_2) &= u \\
k(y_1 - y_2) &= m\ddot{y}_2 + \beta\dot{y}_2
\end{aligned}
$$

where k, β and m are positive constants denoting the spring, damper coefficients and the mass, respectively. The gravity force is neglected.

1. Write the transfer function from u to y_2, named $T_2(s)$, identify the high frequency gain and poles of $T_2(s)$

2. Assume only $y_1(t), y_2(t)$ can be measured at each time instant and $u(t) \in \mathcal{L}_\infty$, design a gradient-based (instantaneous cost function) online estimator to identify the unknown parameters k, and prove that your estimates $\hat{k}(t)$ is an $\mathcal{L}_\infty$ signal.

3. Assume only $y_1(t), y_2(t)$ can be measured at each time instant and $u(t) \in \mathcal{L}_\infty$. Design a pure least-square estimator to identify the unknown parameter β and m. Choose a proper input signal u such that the estimates $\hat{m}$ and $\hat{\beta}$ can asymptotically converge to their true values. Explain your choice.

4. Say we have obtained that $m = \beta = 1$ and $k = 2$. Set $u = -\dot{y}_1 + v(t)$ and let $x := [y_2\ ,\ \dot{y}_2\ ,\ y_1]^\top$, $v(t)$, $e = y_2(t)$ denote the state, input, output of the spring-mass-damper system. Identify the value of matrices A, B, C, D in the following state-space realization

$$
\begin{cases} \dot{x} = Ax + Bv \\ e = Cx + Dv \end{cases} \tag{2}
$$

5. Consider the system model obtained in Question 4, assume only y_2 can be directly measured, design a proper observer-based feedback controller in the form of $v(t) = f(e)$, such that the closed-loop poles are assigned to the roots of

$$A^*(s) = s^3 + 4s^2 + 7s + 4 = 0$$

while the observation error will exponentially decay to zero at the rate of $-1, -2, -3$.

4 Adaptive Controller Desgin(40 points)

Consider a second-order system

$$\dot{x} = Ax + B[u(t) + d(t)], \quad x(0) = x_0 \in \mathbb{R}^2$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

with ζ, ω and b are unknown positive constants, and

$$d(t) = \alpha x^\top x + \beta$$

is a state-dependent disturbance with unknown constants α and β.

1. Given state x available and assume ζ, ω, b and α, β are all *known* for now, design a full-state feedback model reference controller $u(t) = u^*(t)$ such that the closed-loop trajectories are bounded and state x **exponentially** track state x_m of the following reference model

   $$\dot{x}_m = A_m x_m + B_m r, \qquad x_m(0) = 0 \in \mathbb{R}^2$$

   where r is a uniformly bounded reference signal and

   $$A_m = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \quad B_m = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

   (Hint, the control law should be a function of x and r)

2. Prove the effectiveness of the controller you just designed via the Lyapunov equation of A_m. (Hint: you need to proof boundedness and exponential convergence)

3. Now, given ζ, ω, b and α, β are all *unknown*(note that $sgn(b) > 0$ is known), design **a direct model reference controller with an unnormalized adaptive law** to achieve the same control objectives stated in Question 1(Hint: invoke the idea of SPR- Lyapunov-based design, but don't be too obsessed with MYK lemma since we have access to the fullstate x in this problem.)

4. Demonstrate the effectiveness of your design in Question 3 by showing the boundedness of input signal and zero-convergence of the tracking error $e := x_m - x$.